

THE KING WHO LOVES CHESS

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ABSTRACT. I first heard this tale as an undergraduate in an analysis class. The first half of the story, with an appropriate ending, is suitable for students who are having difficulty understanding exponential growth. The entire story reminds beginning analysis students that sequences—actually, series—do not behave as we might imagine.

1. THE KING AND THE PEASANT

There was once lived a great King who loved chess. The King challenged everyone in his kingdom to a game of chess and took pride in winning every game of chess he sat down to play. He played chess with nobles, knights and peasants.

The last person in the kingdom to come see the King for a game of chess was also the poorest person in the kingdom. Somehow, the poor peasant became the first person to ever win a game of chess against the King. The King was sincerely surprised and humbled.

The King told the peasant, “You have earned any prize that you wish. Anything in my kingdom that you desire belongs to you. What will you have?”

The peasant replied, “Your highness, my family is very poor and hungry. I wish to feed them. The amount of grain I want follows a pattern on the chess board. Corresponding to the first square in the chess board, I would like 1 grain of wheat. Corresponding to the second square in the chess board, I would like 2 grains of wheat. I would like to continue in in this manner—doubling the previous quantity corresponding to each square on the chess board.”

The King replied, “But there are only 64 squares on the chess board!” And the King mumbled to himself, “You may not be as intelligent as I thought.” He asked the peasant, “Are you sure that is what you want?”

The peasant told the King, “I am sure that is more than enough grain to feed my entire family. This is exactly the amount of grain I would like.”

2. THE KING AND THE ROYAL MATHEMATICIAN

The King told the peasant that his Royal Mathematician would calculate the exact amount of grain that would be paid to the peasant. They agreed to call this quantity S . Once the peasant left the castle, the King asked the Royal Mathematician to calculate the quantity S .

“Actually, your Highness, we do not have enough grain in the entire kingdom to pay the peasant,” said the Royal Mathematician.

The King thought that this might be some kind of mistake. After all, the kingdom was very opulent and even the poor were not starving. Perhaps the mathematician could explain . . .

“The quantity for the first few squares is small, 1, 2, 4, 8, 16, 32, 64, . . . But each time we are doubling the amount from the previous square. The quantity that corresponds to the 64th square is

$$2^{63} \sim 9 \text{ quintillion!!!}$$

It would take several years to produce that much grain,” explained the Royal Mathematician.

The King was shocked, and again, humbled. **And then the King felt frustrated.** He was certainly not going to be outsmarted by a peasant. There must be a way out of this situation. If there is, his Royal Mathematician would figure it out. After all, mathematicians know when a solution to a problem exists. If a solution exists, it is the task of the Royal Mathematician to find it.

“Royal Mathematician, it is your job to get me out of this mess. I will be in my Royal Chambers. If you cannot figure this out, there will be consequences,” warned the King.

3. THE ROYAL MATHEMATICIAN AND THE PEASANT

The next day the peasant returned to the castle to pick up his prize. He was greeted by the Royal Mathematician who had a contract that the peasant was to sign.

The Royal Mathematician asked the peasant, “You asked for the quantity S which is obtained by summing the quantities, 1, 2, 4, 8, 16, etc. until we reach the quantity 2^{63} , correct?”

“That is correct,” replied the peasant.

“Would you agree that a short hand way of denoting this quantity is,

$$\sum_{n=0}^{63} 2^n,$$

where I have used summation notation?” asked the Royal Mathematician.

“Certainly,” replied the peasant.

“Before we sign the contract, I am prepared to offer you a better prize. Why stop at $n = 63$? We could allow n to go to infinity! That is, would you agree that

$$S = \sum_{n=0}^{\infty} 2^n,$$

is a more appealing prize? You could have this, if you prefer,” offered the Royal Mathematician.

That got the peasant's attention. That peasant thought to himself, *that should be enough grain for generations*, and agreed to the quantity. The Royal Mathematician was delighted. This is what the Royal Mathematician showed the peasant.

$$\begin{aligned} S &= \sum_{n=0}^{\infty} 2^n \\ &= 1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots \\ &= 1 + 2(1 + 2 + 4 + 8 + 16 + 32 + \dots) \\ &= 1 + 2S \end{aligned}$$

The Royal Mathematician told the peasant, "In fact, the quantity $S = -1$, and that means you owe the King one grain of wheat."

The peasant was devastated and confused. What had just happened? Certainly something was wrong. **Had the Royal Mathematician cheated?**

The Royal Mathematician was also confused. He should have paid more attention to his Analysis Professor. But he was relieved that he didn't have to suffer any consequences. For the rest of his life, the Royal Mathematician felt guilt. He should have stopped the King before making any wild promises to the peasant.

The King continues to love chess but is much more careful about quantities promised. He always double checks with his Royal Mathematician. As for the peasant, he encouraged his children to study mathematics. He is sure the Royal Mathematician cheated and hopes that one of his children can prove it one day. $\square$

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